Obtaining the elementary segments and compound segments.

Gonzalo Vilches Urcelay

2023

The following ten pages are a demonstration of the basic concepts that we use in the proof of the Collatz conjecture, that is, elementary segments and compound segments.

This document can be divided into two central points:

First, we will see an algebraic method that allows us to deduce the elementary segments, which are the basic components of the compound segments. The second point concerns the assembly of the composite segments and the differentiation of those assemblies that are applicable to the conjecture from those that are not. This distinction is necessary because if there are composite segments that are obtained by applying the function g two or more times in a row or segments that refute the conjecture through divergences, these would not belong to the theoretical framework of the Collatz conjecture. You can read this in the document “Proof of the Collatz conjecture using the digital root operator” on my website. Anyway, we will see it with some examples in the last pages of this document.

Now let's start by demonstrating the eighteen elementary segments, which are the basic components of the composite segments.

A) Demonstration of the elementary segments.

Let us first remember that the subsets of A_s are the following,

$$A_s = \{A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8,\ A_9\}.$$

Each of the nine subsets of A_s is composed of both even and odd elements. To see this, let's look at the elements of the subset A_1 has:

$$A_1 = \{1 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{1,\ 10\ ,19,\ 28,\ \ldots\}.$$

(1 odd, 10 even, 19 odd, 28 even, …).

Let us also see the elements that conform the second subset of A_s:

$$A_2 = \{2 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{2,\ 11\ ,20,\ 29,\ \ldots\}.$$

(2 even, 11 odd, 20 even, 29 odd, …).

1.- Demonstration of elementary segments one and ten:

Now, if we want to refer to the even elements and the odd elements separately from the subset A_1 then we can do it as follows:

$$A_{evens} = \{1 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{10,\ 28,\ 46,\ 64,\ \ldots\}.$$

$$A_{odds} = \{1 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{1,\ 19,\ 37,\ 55,\ \ldots\}.$$

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_1$, when divided by two, will give each of the elements of A_5:

$$\frac{A_{evens}}{2} = \frac{1+9\cdot(2n+1)}{2} = 5 + 9 \cdot n.$$

In a more visual way, although not formally:

$$A_5 = \frac{\{10, 28, 46, 64, \ldots\}}{2} = \{5,\ 14,\ 23,\ 32,\ \ldots\}.$$

On the other hand, if we consider the elements of $A_{odds} \subset A_1$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_4, which will also be even. We can see this clearly in the following equation:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [1 + 9 \cdot (2n)] + 1 = 4 + 9 \cdot (6n).$$

With these operations we can guarantee the following:

1) When dividing an even number that belongs to A_1 by two, we will always obtain a natural number from the subset A_5.

   $A_1 \rightarrow A_5$ (First elementary segment).

2) When multiplying by three and adding one to an odd number that belongs to A_1 we will always obtain a natural number from the subset A_4.

   $A_1 \mapsto A_4$ (Elementary segment number ten).

2.- Demonstration of elementary segments two and eleven:

If it has not been clear, in the following I will dedicate myself to demonstrating each of the elementary segments, starting with the elements of subset A_2 whose components also alternate between even and odd:

$$A_2 = \{2 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{2,\ 11,\ 20,\ 29,\ \ldots\}.$$

(2 even, 11 odd, 20 even, 29 odd, …).

If we want to refer to the even elements and the odd elements separately, then we can do it as follows:

$$A_{evens} = \{2 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{2,\ 20,\ 38,\ 56,\ \ldots\}.$$

$A_{odds} = \{2 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{11,\ 29,\ 47,\ 65,\ \ldots\}$.

In this way, we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_2$ when divided by two will give each of the elements of A_1:

$\frac{A_{evens}}{2} = \frac{2+9\cdot(2n)}{2} = 1 + 9 \cdot n$.

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_2$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_7, which will also be even. We can see this clearly in the following equations:

$3 \cdot (A_{odds}) + 1 = 3 \cdot [2 + 9 \cdot (2n + 1)] + 1 = 7 + 9 \cdot (6n + 3)$.

After these operations, we can guarantee the following:

1) When dividing an even number that belongs to A_2 we will always obtain a natural number that belongs to A_1.

   $A_2 \to A_1$ (Second elementary segment).

2) When multiplying by three and adding one to an odd number that belongs to A_2, we will always obtain a natural number that belongs to the subset A_7.

   $A_2 \mapsto A_7$ (Elementary segment number eleven),

3.- Demonstration of the elementary segments three and twelve.

To obtain elemental segment number three and twelve, we will do the same procedure that we have applied on the previous pages.

We start from subset A_3, whose elements alternate between even and odd numbers:

$A_3 = \{3 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{3,\ 12,\ 21,\ 30,\ \ldots\}$.

If we want to refer to the even elements and the odd elements separately, we will do so using the following expressions:

$A_{evens} = \{3 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{12,\ 30,\ 48,\ 66,\ \ldots\}$.

$A_{odds} = \{3 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{3,\ 21,\ 39,\ 57,\ \ldots\}$.

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_3$ when divided by two will give each of the elements of A_6:

$\frac{A_{evens}}{2} = \frac{3+9\cdot(2n+1)}{2} = 6 + 9 \cdot n$.

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_3$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_1, which will also be even. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [3 + 9 \cdot (2n)] + 1 = 1 + 9 \cdot (6n + 1).$$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_3 we will always obtain a natural number that belongs to A_6.

   $A_3 \rightarrow A_6$ (Elementary segment number three).

2) When multiplying by three and adding one to an odd number that belongs to A_3, we will always obtain a natural number that belongs to the subset A_1.

   $A_3 \mapsto A_1$ (Elementary segment number twelve).

4.- Demonstration of the elementary segments number four and fifteen.

We start with subset A_4 observing that it is composed of even elements and odd elements:

$$A_4 = \{4 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{4,\ 13,\ 22,\ 31,\ \dots\}.$$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$$A_{evens} = \{4 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{4,\ 22,\ 40,\ 58\ \dots\}.$$

$$A_{odds} = \{4 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{13,\ 31,\ 49,\ 67\ \dots\}.$$

In this way, we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_4$ when divided by two will give each of the elements of A_2:

$$\frac{A_{evens}}{2} = \frac{4+9\cdot(2n)}{2} = 2 + 9 \cdot n.$$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_4$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_4, which will also be even. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [4 + 9 \cdot (2n + 1)] + 1 = 4 + 9 \cdot (6n + 4).$$

After these operations, we can guarantee the following statements:

1) When dividing an even number that belongs to A_4 we will always obtain a natural number that belongs to A_2:

$A_4 \rightarrow A_2$ (Elementary segment number four).

2) When multiplying by three and adding one to an odd number that belongs to A_4, we will always obtain a natural number that belongs to the subset A_4:

$A_4 \mapsto A_4$ (Elementary segment number thirteen).

5.- Demonstration of elementary segments five and fourteen:

We start from subset A_5 observing that it is composed of even elements and odd elements:

$$A_5 = \{5 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{5,\ 14,\ 23,\ 32,\ \dots\}.$$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$$A_{evens} = \{5 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{14,\ 32,\ 50,\ 68,\ \dots\}.$$

$$A_{odds} = \{5 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{5,\ 23,\ 41,\ 59,\ \dots\}.$$

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset A_5 when divided by two will give each of the elements of A_7:

$$\frac{A_{evens}}{2} = \frac{5+9\cdot(2n+1)}{2} = 7 + 9 \cdot n.$$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_5$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_7, which will also be even. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [5 + 9 \cdot (2n)] + 1 = 7 + 9 \cdot (6n + 4).$$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_5 we will always obtain a natural number that belongs to A_7.

$A_5 \rightarrow A_7$ (Elementary segment number five).

2) When multiplying by three and adding one to an odd number that belongs to A_5, we will always obtain a natural number that belongs to the subset A_7.

$A_5 \mapsto A_7$ (Elementary segment number fourteen).

6.- Demonstration of the elementary segments number six and fifteen.

We start from subset A6 observing that it is composed of both even and odd elements:

$A_6 = \{6 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{6,\ 15,\ 24,\ 33,\ \dots\}.$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$A_{evens} = \{6 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{6,\ 24,\ 42,\ 60,\ \dots\}.$

$A_{odds} = \{6 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{15,\ 33,\ 51,\ 69,\ \dots\}.$

In this way, we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_6$ when divided by two will give each of the elements of A_3:

$\frac{A_{evens}}{2} = \frac{6+9\cdot(2n)}{2} = 3 + 9 \cdot n.$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_6$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_1, which will also be even. We can see this clearly in the following expression:

$3 \cdot (A_{odds}) + 1 = 3 \cdot [6 + 9 \cdot (2n + 1)] + 1 = 1 + 9 \cdot (6n + 5).$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_6 we will always obtain a natural number that belongs to A_3.

   $A_6 \to A_3$ (Elementary segment number six).

2) When multiplying by three and adding one to an odd number that belongs to A_6, we will always obtain a natural number that belongs to the subset A_1.

   $A_6 \mapsto A_1$ (Elementary segment number fifteen).

7.- Demonstration of the elementary segments seven and sixteen:

We start from subset A_7 observing that it is composed of both even and odd elements:

$A_7 = \{7 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{7,\ 16,\ 25,\ 34,\ \dots\}.$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$A_{evens} = \{7 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{16,\ 34,\ 52,\ 70,\ \dots\}.$

$A_{odds} = \{7 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{7,\ 25,\ 43,\ 61,\ \dots\}.$

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{odds} \subset A_7$ when divided by two will give each of the elements of A_8:

$$\frac{A_{evens}}{2} = \frac{7+9\cdot(2n+1)}{2} = 8 + 9 \cdot n.$$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_7$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_4, which will also be pairs. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot\ [7 + 9 \cdot (2n)] + 1 = 4 + 9 \cdot (6n + 2).$$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_7 we will always obtain a natural number that belongs to A_8.

   $A_7 \rightarrow A_8$ (Elementary segment number seven).

2) When multiplying by three and adding one to an odd number that belongs to A_7 we will always obtain a natural number that belongs to the subset A_4.

   $A_7 \mapsto A_4$ (Elementary segment number sixteen).

8.- Demonstration of the elementary segments eight and seventeen:

We start from subset A_8 observing that it is composed of both even and odd elements:

$$A_8 = \{8 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{8,\ 17,\ 26,\ 35,\ \ldots\}.$$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$$A_{evens} = \{8 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{8,\ 26,\ 44,\ 62\ \ldots\}.$$

$$A_{odds} = \{8 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{17,\ 35,\ 53,\ 71\ \ldots\}.$$

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_8$ when divided by two will give each of the elements of A_4:

$$\frac{A_{evens}}{2} = \frac{8+9\cdot(2n)}{2} = 4 + 9 \cdot n.$$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_8$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_7, which will also be even. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [8 + 9 \cdot (2n + 1)] + 1 = 7 + 9 \cdot (6n + 5).$$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_8 we will always obtain a natural number that belongs to A_4.

   $A_8 \to A_4$ (Elementary segment number eight).

2) When multiplying by three and adding one to an odd number that belongs to A_8, we will always obtain a natural number that belongs to the subset A_7.

   $A_8 \mapsto A_7$ (Elementary segment number seventeen).

9.- Demonstration of the elementary segments nine and eighteen:

We start from subset A_9 observing that it is composed of both even and odd elements:

$$A_9 = \{9 + 9 \cdot n \mid n \in \mathbb{N}_0\} = \{9,\ 18,\ 27,\ 36,\ \dots\}.$$

If we want to refer to the even and odd elements separately, we can do so using the following expressions:

$$A_{evens} = \{9 + 9 \cdot (2n + 1) \mid n \in \mathbb{N}_0\} = \{18,\ 36,\ 54,\ 72,\ \dots\}.$$

$$A_{odds} = \{9 + 9 \cdot (2n) \mid n \in \mathbb{N}_0\} = \{9,\ 27,\ 45,\ 63\ \dots\}.$$

In this way we can operate algebraically with the respective subsets, and we can easily verify that all the elements of the subset $A_{evens} \subset A_9$ when divided by two will give each of the elements of A_9:

$$\frac{A_{evens}}{2} = \frac{9+9\cdot(2n+1)}{2} = 9 + 9 \cdot n.$$

On the other hand, if we consider the elements of the subset $A_{odds} \subset A_9$ and, as the conjecture says, we multiply by three and add one to each of the elements of the subset, we will always obtain elements of the subset A_1, which will also be even. We can see this clearly in the following expression:

$$3 \cdot (A_{odds}) + 1 = 3 \cdot [9 + 9 \cdot (2n)] + 1 = 1 + 9 \cdot (6n + 3).$$

After these operations we can guarantee the following statements:

1) When dividing an even number that belongs to A_9 we will always obtain a natural number that belongs to the same set, A_9.

   $A_9 \to A_9$ (Elementary segment number nine).

2) When multiplying by three and adding one to an odd number that belongs to A_9, we will always obtain a natural number that belongs to the subset A_1.

   $A_9 \mapsto A_1$ (Elementary segment number eighteen).

B) Combination of elementary segments.

When combining elementary segments in order to obtain the so-called composite segments, we must take the following into account:

(a) We can only combine two elemental segments if the first component of one of them coincides with the second component of the other elemental segment.
(b) After multiplying by three and adding one to any number, the next operation applied is always dividing by two.

As an example, we will see how to construct the following two composite segments from the elementary segments, and we will see which of them is applicable to the conjecture and which is not:

$$\eta : A_7 \to A_8 \to A_4 \mapsto A_4 \to A_2.$$

$$\mu : A_4 \to A_2 \to A_1 \mapsto A_4 \mapsto A_4.$$

The composite segment η is created from the following elementary segments,

$$A_7 \to A_8 \ ; \ A_8 \to A_4 \ ; \ A_4 \mapsto A_4 \ ; \ A_4 \to A_2.$$

and it is applicable to the conjecture because it considers sentences (a) and (b).

On the other hand, the composite segment μ is created from the following elementary segments,

$$A_4 \to A_2 \ ; \ A_2 \to A_1 \ ; \ A_1 \mapsto A_4 \ ; \ A_4 \mapsto A_4.$$

However, it is not applicable to the conjecture since, even that satisfies sentence (a), does not satisfy sentence (b). This is so because, when going from $A_1 \mapsto A_4$ we have multiplied by three and added one, and then we have combined it with the segment $A_4 \mapsto A_4$, which is also obtained by multiplying by three and adding one. Therefore, the composite segment μ is not applicable to the conjecture since in no composite segment two multiplications can occur in a row (or the same thing, applying the function g twice in a row).

Brief Summary:

From the elementary segments, we can obtain all the existing composite segments. As we have seen, not all compound segments are segments that hold within the theoretical framework of the Collatz conjecture.

Composite segments can be classified according to whether they are applicable to the conjecture or not:

Applicable to the conjecture:	Not applicable to the conjecture:
Numerical flows of all natural numbers not including zero.	Divergences. Impossible segments (break at least one of the two conditions set out on the previous page).